ON ANALYTIC FAMILIES OF OPERATORS

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The classical Riesz-Thorin interpolation theorem [6] was extended by Hirschman [2] and Stein [5] to analytic families of operators. We recall the notions:

Let $F(z)$, $z = x + iy$, be analytic in $0 < \mathrm{Re}\, z < 1$ and continuous in $0 \leqq \mathrm{Re}\, z \leqq 1$. $F(z)$ is said to be of admissible growth iff

$$\sup_{0 \leqq x \leqq 1} \log |F(x+iy)| \leqq Ae^{a|y|} \text{ where } a < \pi.$$

The significance of this notion is in the following lemma due to Hirschman [2]:

LEMMA. *If $F(z)$ is of admissible growth and if* $\log|F(it)| \leqq a_0(t)$, $\log|F(i+it)| \leqq a_1(t)$ *then* $\log|F(\theta)| \leqq \int_{-\infty}^{\infty} P_0(\theta, t)a_0(t)dt + \int_{-\infty}^{\infty} P_1(\theta, t)a_1(t)dt$ *where $P_i(\theta, t)$ are the values of the Poisson kernel for the strip, on* $\mathrm{Re}\, z = 0$, $\mathrm{Re}\, z = 1$.

We next define analytic families of linear operators: Let (M, μ) (N, ν) be two measure spaces. Let $\{T_z\}$ be a family of linear operators indexed by z, $0 \leqq \mathrm{Re}\, z \leqq 1$ so that for each z, T_z is a mapping of simple functions on M to measurable functions on N. $\{T_z\}$ is called an analytic family iff for any measurable set E of M of finite measure, for almost every $y \in N$, the function $\phi_y(z) = T_z(X_E)(y)$ is analytic in $0 < \mathrm{Re}\, z < 1$, continuous in $0 \leqq \mathrm{Re}\, z \leqq 1$. The analytic family is of admissible growth iff for almost every $y \in N$, $\phi_y(z)$ is of admissible growth.

We finally recall the notion of $L(p, q)$ spaces. An exposition of these spaces can be found in Hunt [3].

Let f be a complex valued measurable function defined on a σ-finite measure space (M, μ). μ is assumed to be non-negative. We assume that f is finite valued a.e., and denoting

$$E_y = \{x / |f(x)| > y\}, \quad \lambda_f(y) = \mu(E_y),$$

we assume also that for some $y > 0$, $\lambda_f(y) < \infty$. We define

$$f^*(t) = \mathrm{Inf}\{y > 0 / \lambda_f(y) \leqq t\}.$$

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This is called the non decreasing rearrangement of f. We define

$$\| f \|^*_{pq} = \begin{cases} \left(\frac{q}{p} \int_0^\infty t^{q/p}[f^*(t)]^q \frac{dt}{t}\right)^{1/q} & 0 < p < \infty, 0 < q < \infty \\ \operatorname*{Sup}_{0<t} t^{1/p} f^*(t) & 0 < p \leqq \infty,\ q = \infty \end{cases}$$

and $L(p,q) = \{f / \| f \|^*_{pq} < \infty\}$. For $p = q$ these are the usual L^p spaces, while for $q = \infty$ we have the so-called weak L^p spaces, i.e., the spaces of functions which satisfy $\lambda_f(y) \leqq C/y^p$.

Many of the proofs are simplified if we make use of the following auxiliary function:

For any $0 < r \leqq 1$ $r \leqq q$, $r < p$ we define

$$f^{**}(t) = f^{**}(t,r) = \begin{cases} \operatorname{Sup} \left\{ \left(\frac{1}{\mu(E)} \int_E |f(x)|^r d\mu(x)\right)^{1/r} / \mu(E) > t \right\}, & t < \mu(M) \\ \left(\frac{1}{t} \int_M |f(x)|^r d\mu(x)\right)^{1/r}, & \mu(M) \leqq t \end{cases}$$

Since f^* is non-increasing we have $(f^*)^{**}(t) = (1/t \int_0^t [f^*(u)]^r du)^{1/r}$, and since f^{**} is continuous from the right and non-increasing, we have $(f^{**})^* = f^{**}$. We can show

$$f^*(t) \leqq f^{**}(t) \leqq (f^*)^{**}(t)$$

which yields

$$\| f \|^*_{pq} \leqq \| f^{**} \|^*_{pq} \leqq \| (f^*)^{**} \|^*_{pq}$$

while from Hardy's inequality [3, pp. 256)] one has

$$\| (f^*)^{**} \|^*_{pq} \leqq \left(\frac{p}{p-r}\right)^{1/r} \| f \|^*_{pq}$$

and so the topologies defined on $L(p,q)$ by all these functions are equivalent. We denote $\| f^{**} \|^*_{pq} = \| f \|_{pq}$. We can now prove the following theorem:

THEOREM. *If $\{T_z\}$ is an analytic family of linear operators, which is of admissible growth, then if for all simple functions*

$$\text{(1)} \qquad \| T_{it} f \|^*_{\bar{p}_0 \bar{q}_0} \leqq A_0(t) \| f \|^*_{p_0 q_0}$$

$$\text{(2)} \qquad \| T_{1+it} f \|^*_{\bar{p}_1 \bar{q}_1} \leqq A_1(t) \| f \|^*_{p_1 q_1},$$

where $\log A_i(t) \leqq Ae^{a|t|}$ $a<\pi$, *then for* $0<\theta<1$,

$$\frac{1}{\bar{p}} = \frac{1-\theta}{\bar{p}_0}+\frac{\theta}{\bar{p}_1} \qquad \frac{1}{\bar{q}} = \frac{1-\theta}{\bar{q}_0}+\frac{\theta}{\bar{q}_1}$$

$$\frac{1}{p} = \frac{1-\theta}{p_0}+\frac{\theta}{p_1} \qquad \frac{1}{q} = \frac{1-\theta}{q_0}+\frac{\theta}{q_1}$$

we have for all simple functions f

$$\| T_\theta f \|^*_{\bar{p}\bar{q}} \leqq BA_\theta \| f \|^*_{pq} \tag{3}$$

where $\log A_\theta \leqq \int_{-\infty}^{\infty} P_0(\theta,t)\log A_0(t)dt + \int_{-\infty}^{\infty} P_1(\theta,t)\log A_1(t)dt$.

The following lemma will be basic in the proof:

Lemma. *Let* $\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}$ $\frac{1}{q}=\frac{1-\theta}{q_0}+\frac{\theta}{q_1}$, *and let f be a simple function,* $\| f\|^*_{pq}=1$. *Then we can find non-negative simple functions* $G_0(x)$, $G_1(x)$ *so that*

$$f(x) = e^{i\arg f(x)}[G_0(x)]^{1-\theta}[G_1(x)]^\theta$$

with $\| G_i(x)\|^*_{p_iq_i} \leqq B$.

Proof. The case $q_0 \neq \infty$, $q_1 \neq \infty$ is done in [3, p. 266]. The proof when one of the q_i, say q_0, is $\neq \infty$, is included implicitly there: One takes

$$h_0(t) = t^{-1/p_0} \qquad h_1(t) = [(f^*)^{**}]^{q/q_1}t^{1/q_1(q/p-q_1/p_1)},$$

and the proof proceeds as in [3].

When both $q_i=\infty$, write $h_i(t) = (f^*(t))^{p/p_i}$, Clearly

$$\|h_i(t)\|^*_{p_i\infty} = \sup_{0<t} t^{1/p_i}[f^*(t)]^{p/p_i} = \sup_{0<t}[t^{1/p}f^*(t)]^{p/p_i} = (\|f\|^*_{p\infty})^{p/p_i} = 1.$$

$h_i(t)$ are non-increasing step functions, continuous from the right and so are fit to serve as rearrangements of simple functions. The sets of constancy of h_i are the sets of constancy of f^* and so correspond to the sets of constancy of f.

$G_i(x)$ are now defined on the sets of constancy of f, and have the same values there as $h_i(t)$ have on the corresponding sets. Clearly $G_i^* = h_i$ and so $\| G_i\|^*_{p_i\infty}=1$. Finally, since $f^*(t) = h_0^{1-\theta}(t)h_1^\theta(t)$, we have $f(x) = e^{i\arg f(x)}[G_0(x)]^{1-\theta}[G_1(x)]^\theta$.

Let us now proceed with the proof of the theorem. Let f be a simple function, $\|f\|^*_{pq}=1$. Define

$$F(x,z) = e^{i\arg f(x)}[G_0(x)]^{1-z}[G_1(x)]^z.$$

Since $G_i(x)$ are simple and non-negative, $T_zF(\cdot, z)(y)$ is for almost every $y \in N$ an analytic function of z in $0 < \operatorname{Re} z < 1$, continuous in $0 \leqq \operatorname{Re} z \leqq 1$, and of admissible growth. Writing $T_zF(y,z)$ for $T_zF(\cdot,z)(y)$ we therefore have

$$\text{(1)} \qquad \log|T_\theta F(y,\theta)| \leqq \int_{-\infty}^{\infty} P_0(\theta,t)\log|T_{it}F(y,it)|\,dt + \int_{-\infty}^{\infty} P_1(\theta,t)\log|T_{1+it}F(y,1+it)|\,dt.$$

Note: $T_\theta F(y,\theta) = (T_\theta f)(y)$.

Taking exponentials of both sides of (1) we get

$$\text{(2)} \qquad |T_\theta f(y)| \leqq \left[\left\{\exp\left(\frac{1}{1-\theta}\int_{-\infty}^{\infty} P_0(\theta,t)\log|T_{it}F(y,it)|^r dt\right)\right\}^{1/r}\right]^{1-\theta} \times \left[\left\{\exp\left(\frac{1}{\theta}\int_{-\infty}^{\infty} P_1(\theta,t)\log|T_{1+it}F(y,1+it)|^r dt\right)\right\}^{1/r}\right]^{\theta}.$$

Since $\|(1-1/n)T_\theta f\|_{\bar p \bar q}^{*} \nearrow \|T_\theta f\|_{\bar p \bar q}^{*}$ we can assume that we have strict inequality in (2), for every y.

Denote by E_k the set of all points y so that (2) holds when the integrations are performed over $|t| < K_1$ for all $k < K_1$. Clearly then (since we assume strict inequality in (2)) $E_K \nearrow N$ and so $\|T_\theta f\|_{pq}^{*} = \lim_{k\to\infty} \|\chi_{E_k} T_\theta f\|_{pq}^{*}$.

We can therefore assume

$$\text{(3)} \qquad |T_\theta f| \leqq \left[\left\{\exp\left(\frac{1}{1-\theta}\int_{-k}^{k} P_0(\theta,t)\log|T_{it}F(y,it)|^r dt\right)\right\}^{1/r}\right]^{1-\theta} \times \left[\left\{\exp\left(\frac{1}{\theta}\int_{-k}^{k} P_1(\theta,t,)\log|T_{1+it}F(y,1+it)|^r dt\right)\right\}^{1/r}\right]^{\theta}$$

Denote $l_{ik} = \int_{-k}^{k} P_i(\theta, t)dt$. Since $P_i(\theta, t) \geqq 0$, $\int_{-\infty}^{\infty} P_0(\theta,t)dt = 1-\theta$, $\int_{-\infty}^{\infty} P_1(\theta,t)dt = \theta$ we have $l_{0k} \nearrow 1-\theta$ $l_{1k} \nearrow \theta$, and we have

$$\text{(4)} \qquad |T_\theta f| \leqq \left[\left\{\exp\left(\frac{1}{l_{0k}}\int_{-k}^{k} P_0(\theta,t)\log|T_{it}F(y,it)|^r dt\right)\right\}^{1/r}\right]^{1-\theta} \times \left[\left\{\exp\left(\frac{1}{l_{1k}}\int_{-k}^{k} P_1(\theta,t)\log|T_{1+it}F(y,1+it)|^r dt\right)\right\}^{1/r}\right]^{\theta}$$

Using Jensen's inequality we get:

$$|T_\theta f| \leqq \left[\left\{\frac{1}{l_{0k}}\int_{-k}^{k} P_0(\theta,t)\,|T_{it}F(y,\ it)|^r dt\right\}^{1/r}\right]^{1-\theta}$$
$$\times \left[\left\{\frac{1}{l_{1k}}\int_{-k}^{k} P_1(\theta,t)\,|T_{1+it}F(y,1+it)|\,dt\right\}^{1/r}\right]^{\theta}$$

Denote

$$H_0(y) = \left[\frac{1}{l_{0k}}\int_{-k}^{k} P_0(\theta,t)\,|T_{it}F(y,\ it)|^r dt\right]^{1/r},$$
$$H_1(y) = \left[\frac{1}{l_{1k}}\int_{-k}^{k} P_1(\theta,t)\,|T_{1+it}F(y,\ _1+it)|^r dt\right]^{1/r};$$

and then $|T_\theta f| \leqq [H_0(y)]^{1-\theta}[H_1(y)]^\theta$.

Hölder's inequality implies $T_\theta f^{**}(v) \leqq [H_0^{**}(v)]^{1-\theta}[H_1^{**}(v)]^\theta$, and then

(5) $$\|T_\theta f\|^*_{\bar p\bar q} \leqq B\,\|H_0\|^{1-\theta}_{\bar p_0\bar q_0}\,\|H_1\|^{\theta}_{\bar p_1\bar q_1}$$

By Fubini's theorem

$$H_0^{**}(v) \leqq \left(\frac{1}{l_{0k}}\int_{-k}^{k} P_0(\theta,t)\,|T_{it}F^{**}(v,it)|^r dt\right)^{1/r}$$

and so for $\bar q_0 < \infty$

(6) $$\|H_0\|_{\bar p_0\bar q_0} \leqq \left[\frac{\bar q_0}{\bar p_0}\int_0^\infty \left[\frac{1}{l_{0k}}\int_{-k}^{k} P_0(\theta,t)\,|T_{it}F^{**}(v,\ it)|^r dt\right]^{\bar q_0/r} v^{\bar q_0/\bar p_0}\frac{dv}{v}\right]^{1/\bar q_0}$$

while for $\bar q_0 = \infty$

(6′) $$\|H_0\|_{\bar p_0\infty} \leqq \sup_{0<v} v^{1/\bar p_0}\left(\frac{1}{l_{0k}}\int_{-k}^{k} P_0(\theta,t)\,|T_{it}F^{**}(v,\ it)|^r dt\right)^{1/r}$$

the proof in the second case is similar to the proof when $\bar q_0 < \infty$. We shall leave it to the reader and continue from (6).

Using the integral form of Minkowski's inequality, we get from (6):

$$\|H_0\|_{\bar q_0\bar p_0} \leqq \left(\frac{1}{l_{0k}}\int_{-k}^{k} P_0(\theta,t)\left[\frac{\bar q_0}{\bar p_0}\int_0^\infty |T_{it}F^{**}(v,it)|^{\bar q_0} v^{\bar q_0/\bar p_0}\frac{dv}{v}\right]^{r/\bar q_0} dt\right)^{1/r}$$
$$= \left(\frac{1}{l_{0k}}\int_{-k}^{k} P_0(\theta,t)\,\|T_{it}F(\cdot,\ it)\|^r_{\bar p_0\bar q_0} dt\right)^{1/r}$$
$$\leqq B\left(\frac{1}{l_{0k}}\int_{-k}^{k} P_0(\theta,t)A_0^r(t)\,\|G_0\|^r_{\bar p_0\bar q_0} dt\right)^{1/r}$$
$$\leqq B\left(\frac{1}{l_{0k}}\int_{-k}^{k} P_0(\theta,t)A_0^r(t)dt\right)^{1/r}$$

Similarly

$$\| H_1 \|_{\bar{p}_1\bar{q}} \leqq B \left[\frac{1}{l_{1k}} \int_{-k}^{k} P_1(\theta, t) A_1^r(t) dt \right]^{1/r} \text{ and so from (5):}$$

(7)
$$\| T_\theta f \|^*_{\bar{p}\bar{q}} \leqq B \left\{ \left[\frac{1}{l_{0k}} \int_{-k}^{k} P_0(\theta, t) A_0^r(t) dt \right]^{1/r} \right\}^{1-\theta}$$
$$\times \left\{ \left[\frac{1}{l_{1k}} \int_{-k}^{k} P_1(\theta, t) A_1(t) dt \right]^{1/r} \right\}^{\theta}$$

We let now $r \to 0$ and get:

$$\| T_\theta f \|^*_{\bar{p}\bar{q}} \leqq B \left[\exp\left(\frac{1}{l_{0k}} \int_{-k}^{k} P_0(\theta, t) \log A_0(t) dt \right) \right]^{1-\theta}$$
$$\times \left[\exp\left(\frac{1}{l_{1k}} \int_{-k}^{k} P_1(\theta, t) \log A_1(t) dt \right) \right]^{\theta}$$

Letting now $k \to \infty$ we get:

$$\| T_\theta f \|^*_{\bar{p}\bar{q}} \leqq B \exp\left(\int_{-\infty}^{\infty} P_0(\theta, t) \log A_0(t) dt \right) \exp\left(\int_{-\infty}^{\infty} P_1(\theta, t) \log A_1(t)\, dt \right),$$

and the theorem is proved.

Since $L(p,q)$ are complete, and since for $q < \infty$, simple functions are dense in $L(p,q)$, we can, if $q < \infty$, extend T_θ to all of $L(p,q)$ and get

$$\| T_\theta f \|_{\bar{p}\bar{q}} \leqq BA_\theta \| f \|_{pq}.$$

In the case of a single operator, we can prove the norm inequality for all $f \in L(p,q)$ from the result for simple functions also when $q = 00$. See Hunt [3].

We notice that if $\bar{q}_i = \infty$, $i = 0, 1$, then $\bar{q} = \infty$. Thus from weak type at the endpoints, we get weak type in the segment. For a single operator this is not the best result. From Marcinkiewicz's theorem [7] we get strong type in the open segment from weak type at the endpoints. For a family of operators, however, we cannot improve the result as the following comments indicate.

Muckenhoupt in [4] showed that fractional integral operators

$$D_\lambda f = \int_{E^n} \frac{\Omega(t)}{|t|^{n\lambda}} f(x - t) dt \qquad 0 \leqq \lambda < 1$$

where $\Omega(t) = \Omega\left(\frac{t}{|t|}\right)$ is in $L^{1/\lambda}$ on the unit sphere, can be represented as the values for $z = \lambda$ of an analytic family of operators $\{T_z\}$, of admissible growth, and satisfying:

$$\| T_{it}f \|^*_{\infty,\infty} \leqq \| f \|^*_{1,1}$$

$$\| T_{1+it}f \|^*_{1,\infty} \leqq B_1 \| f \|^*_{1,1} .$$

Applying our interpolation theorem we get

$$\| D_\lambda f \|^*_{1/\lambda,\infty} \leqq B_\lambda \| f \|^*_{1,1} .$$

I.e. D_λ maps $L(1,1)$ into $L(1/\lambda, \infty)$. For a different proof of this result see Zygmund [7].

Take now $n = 1$, $\Omega(t) = 1$, and consider

$$f_n(t) = \begin{cases} n & -\frac{1}{n} < t < 0 \\ 0 & \text{elsewhere.} \end{cases}$$

For positive values of x we have $D_\lambda f_n(x) = \frac{1}{(x+\xi)^\lambda}$ where $\xi = \xi(x)$, $0 < \xi < \frac{1}{n}$. Thus

$$(D_\lambda f_n)^*(t) \geqq \begin{cases} Cn^\lambda & 0 < t < \frac{1}{n} \\ \frac{C}{t^\lambda} & \frac{1}{n} \leqq t . \end{cases} \tag{8}$$

By computing the $L(p,q)$ norms for the functions on the right hand side of (8) we see that they will be uniformly bounded (as they should for $\| f_n \|_{1,1} = 1$) only if $p = 1/\lambda$, $q = \infty$.

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